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Solution of Hyers–Ulam stability problem for generalized Pappus' equation ☆

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Abstract

The purpose of this paper is to solve the stability problem of Ulam for an approximate mapping $f : X \to Y$ of the following generalized Pappus' equation:

$$n^2 Q(x + my) + mn Q(x - ny) = (m + n)\big[n Q(x) + m Q(ny)\big]$$

for all $x, y \in X$ with X and Y linear spaces.

Keywords: Hyers–Ulam stability; Quadratic mapping; Pappus' equation

1. Introduction

In 1940, S.M. Ulam [15] raised a question concerning the stability of group homomorphisms: Let $(G_1, \cdot)$ be a group and let $(G_2, *, d)$ be a metric group with the metric $d(\cdot, \cdot)$.

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Given $\epsilon > 0$, does there exist $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d\big(h(x \cdot y), h(x) * h(y)\big) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d\big(h(x), H(x)\big) < \epsilon$$

for all $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism then there is a true homomorphism near it with small error as much as possible. If the answer is affirmative, we would say the equation of homomorphism $H(x \cdot y) = H(x) * H(y)$ is stable. In 1978 P.M. Gruber [6] imposed the following more general problem: "Suppose a mathematical object satisfies a certain property approximately. Is it then possible to approximate this objects by objects satisfying the property exactly?" This problem is of particular interest in probability theory and in the case of functional equations of different types. First, Ulam's question for approximately additive mappings with constant difference was solved by D.H. Hyers [7] and then generalized by Th.M. Rassias [13] who permitted the Cauchy difference to become unbounded. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. A large list of references can be found, for example, in the papers [5,6,9] and references therein.

It is well known that a mapping f between real vector spaces satisfies the following quadratic functional equation:

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.1}$$

for all x, y if and only if there is a unique symmetric biadditive mapping B such that $f(x) = B(x, x)$ for all x, where B is given by

$$B(x, y) = \frac{1}{4}\big(f(x + y) - f(x - y)\big)$$

(see [1,8]). A stability problem for the quadratic functional equation (1.1) was solved by a lot of authors [4,14]. In particular, J.M. Rassias [11,12] has solved the stability problem of Ulam for the Euler–Lagrange type quadratic functional equation. Furthermore, Jun and Lee [10] have proved the Hyers–Ulam–Rassias stability problem for the pexiderized quadratic equation

$$f(x + y) + g(x - y) = 2h(x) + 2k(y)$$

of mappings f, g, h and k.

Lemma 1.1. *If $m, n \in \mathbb{N}$ are given positive integers and $\triangle ABC$ is a triangle and I and D interior points of the side $\overline{BC}$ with*

$$n|\overline{BI}| = m|\overline{CI}|$$

and $\overline{AD} \perp \overline{BC}$, then the generalized Pappus' identity

$$n^2|\overline{AB}|^2 + mn|\overline{AC}|^2 = (m+n)\big(n|\overline{AI}|^2 + m|\overline{CI}|^2\big) \tag{1.2}$$

holds.

Proof. We consider the case with $m \leqslant n$. Employing the equation $n|\overline{BI}| = m|\overline{CI}|$ and standard results from the classical Euclidean geometry, we get the following geometric identity:

$$\begin{aligned} n^2|\overline{AB}|^2 &= n^2\big(|\overline{AI}|^2 + |\overline{BI}|^2 + 2|\overline{BI}||\overline{ID}|\big) \\ &= n^2|\overline{AI}|^2 + m^2|\overline{CI}|^2 + 2mn|\overline{CI}||\overline{ID}|, \end{aligned}$$

because of the obtuse angle $\angle BIA$ standing opposite to the side $|\overline{AB}|$, as well as the identity

$$mn|\overline{AC}|^2 = mn|\overline{AI}|^2 + mn|\overline{CI}|^2 - 2mn|\overline{CI}||\overline{ID}|,$$

because of the acute angle $\angle AIC$ standing opposite to the side $|\overline{AC}|$ (see Fig. 1). Adding these two new identities, we establish the aforementioned identity (1.2).

Similarly, we obtain the corresponding case with $m \geqslant n$. □

Employing in this paper the above identity (1.2) we introduce the new functional equation

$$n^2Q(x+my) + mnQ(x-ny) = (m+n)\big(nQ(x) + mQ(ny)\big) \tag{1.3}$$

for a mapping $Q: X \to Y$ and for all $x, y \in X$ with X and Y linear spaces. In particular, if $m = n = 1$ in (1.2) then

$$|\overline{AB}|^2 + |\overline{AC}|^2 = 2\big(|\overline{AI}|^2 + |\overline{CI}|^2\big)$$

is the well-known Pappus' identity. For this reason, Eq. (1.3) is called the generalized Pappus' equation. Note that Eq. (1.3) for $m = n = 1$ reduces to the quadratic functional equation (1.1). Let $B: X^2 \to Y$ be a symmetric biadditive mapping. Put $Q(x) := B(x, x)$ for all $x \in X$. Then $Q: X \to Y$ satisfies Eq. (1.3). The mapping Q satisfying (1.3) may be called Pappus type quadratic because the identity

$$n^2(x+my)^2 + mn(x-ny)^2 = (m+n)\big(nx^2 + m(ny)^2\big)$$

holds for all real x, y and $m, n \in \mathbb{N}$, whose geometric interpretation leads to (1.2) on the triangle ABC with $n|\overline{BI}| = m|\overline{CI}|$ for the point I in the side $\overline{BC}$.

In this paper, we are going to investigate the Hyers–Ulam stability problem for the generalized Pappus' equation (1.3).

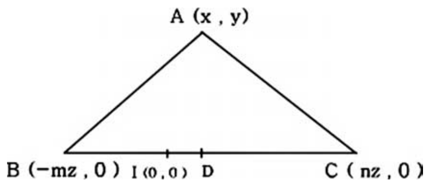

Fig. 1.

2. Stability of (1.3)

We will investigate under what conditions it is then possible to find a true Pappus type quadratic mapping near an approximate Pappus type quadratic mapping with small error.

Lemma 2.1. *Let $Q : X \to Y$ be a Pappus type quadratic mapping satisfying Eq.* (1.3). *Then Q satisfies the equation*

$$Q(x) = \lambda^{-2k} Q(\lambda^k x) \tag{2.1}$$

for all $x \in X$ and all $k \in \mathbb{N}$, where $\lambda := (m+n)/n$.

Proof. In fact, substitution of $x = y = 0$ in (1.3) yields $Q(0) = 0$. Substituting x/n for y in (1.3) one gets the equation

$$n^2 Q\left(\frac{m+n}{n}x\right) = (m+n)^2 Q(x), \quad \text{or}$$
$$Q(\lambda x) = \lambda^2 Q(x) \tag{2.2}$$

for all $x \in X$. Now the basic equation (2.2) with $x \to \lambda^{k-1}x$ yields that the equation

$$Q(\lambda^k x) = \lambda^2 Q(\lambda^{k-1}x) \tag{2.3}$$

holds for all $x \in X$. Moreover, by induction hypothesis with $k \to k-1$ in (2.1) we get that

$$\lambda^{2(k-1)} Q(x) = Q(\lambda^{k-1}x) \tag{2.4}$$

holds for all $x \in X$. Thus Eqs. (2.3), (2.4) imply

$$Q(\lambda^k x) = \lambda^2 Q(\lambda^{k-1}x) = \lambda^2 \lambda^{2(k-1)} Q(x) = \lambda^{2k} Q(x)$$

for all $x \in X$, which completes the proof of Lemma 2.1. □

From now on X and Y will be a vector space and a Banach space, respectively, unless we give any specific reference. Given a mapping $f : X \to Y$ and fixed $m, n \in \mathbb{N}$, we set conveniently

$$D_{m,n} f(x, y) := n^2 f(x+my) + mnf(x-ny) - (m+n)\big(nf(x) + mf(ny)\big)$$

for all $x, y \in X$ and $\lambda := (m+n)/n$.

Theorem 2.2. *Assume that a mapping $f : X \to Y$ satisfies*

$$\big\| D_{m,n} f(x, y)\big\| \leqslant \varphi(x, y) \tag{2.5}$$

for all $x, y \in X$ with an approximate remainder $\varphi : X^2 \to [0, \infty)$, and the series

$$\Phi(x, y) := \sum_{i=0}^{\infty} \frac{\varphi(\lambda^i x, \lambda^i y)}{\lambda^{2i}} \tag{2.6}$$

converges for all $x, y \in X$. there exists a unique Pappus type quadratic mapping $Q : X \to Y$ such that

$$D_{m,n}Q(x,y)=0$$

for all $x, y \in X$ *and*

$$\left\| f(x)-\frac{nf(0)}{m+2n}-Q(x)\right\| \leqslant \frac{1}{(m+n)^2}\Phi\left(x,\frac{x}{n}\right) \tag{2.7}$$

for all $x \in X$*, where*

$$\|f(0)\| \leqslant \inf\left\{\frac{\varphi(x,0)}{m(m+n)}\colon x\in X\right\}.$$

The mapping Q *is given by*

$$Q(x)=\lim_{k\to\infty}\frac{f(\lambda^k x)}{\lambda^{2k}}$$

for all $x \in X$.

Proof. *Step* 1. We show that

$$g(x):=f(x)-\frac{f(0)}{\lambda+1}=f(x)-\frac{nf(0)}{m+2n}$$

satisfies the functional inequality

$$\left\| g(x)-\frac{g(\lambda^k x)}{\lambda^{2k}}\right\| \leqslant \frac{1}{(m+n)^2}\sum_{i=0}^{k-1}\frac{\varphi(\lambda^i x,\lambda^i x/n)}{\lambda^{2i}} \tag{2.8}$$

for all $x \in X$ and all $k \in \mathbb{N}$.

Now letting $y=0$ in (2.5), we have

$$m(m+n)\|f(0)\| \leqslant \varphi(x,0)$$

for all $x \in X$, and hence we get

$$\|f(0)\| \leqslant \inf\left\{\frac{\varphi(x,0)}{m(m+n)}\colon x\in X\right\}.$$

In fact, substituting x/n for y in (2.5) one gets that the inequality

$$\left\|n^2 f(\lambda x)+mnf(0)-(m+n)^2 f(x)\right\| \leqslant \varphi\left(x,\frac{x}{n}\right), \quad \text{or}$$

$$\left\|\frac{f(\lambda x)+mf(0)}{\lambda^2}-f(x)\right\| \leqslant \frac{\varphi(x,x/n)}{(m+n)^2}, \quad \text{or}$$

$$\left\|\frac{f(\lambda x)-f(0)/(\lambda+1)}{\lambda^2}-\left[f(x)-\frac{f(0)}{\lambda+1}\right]\right\| \leqslant \frac{\varphi(x,x/n)}{(m+n)^2},$$

holds for all $x \in X$. Thus we find that the essential inequality

$$\left\| g(x)-\frac{g(\lambda x)}{\lambda^2}\right\| \leqslant \frac{\varphi(x,x/n)}{(m+n)^2} \tag{2.9}$$

holds for all $x \in X$ and hence (2.8) is true for $k = 1$. Replacing now x with λx in (2.9) and dividing it by λ^2, one concludes that

$$\left\| \frac{g(\lambda x)}{\lambda^2} - \frac{g(\lambda^2 x)}{\lambda^4} \right\| \leqslant \frac{\varphi(\lambda x, \lambda x/n)}{(m+n)^2 \lambda^2} \tag{2.10}$$

holds for all $x \in X$. Inequalities (2.9), (2.10) and the triangle inequality yield

$$\left\| g(x) - \frac{g(\lambda^2 x)}{\lambda^4} \right\| \leqslant \frac{1}{(m+n)^2} \left(\varphi\left(x, \frac{x}{n}\right) + \frac{\varphi(\lambda x, \lambda x/n)}{\lambda^2} \right).$$

By induction on $k \in \mathbb{N}$ we assume that (2.8) holds for k and all $x \in X$. Now the basic inequality (2.9) with $x \to \lambda^k x$ yields the inequality

$$\left\| g(\lambda^k x) - \frac{g(\lambda^{k+1} x)}{\lambda^2} \right\| \leqslant \frac{\varphi(\lambda^k x, \lambda^k x/n)}{(m+n)^2},$$

which gives rise to

$$\left\| \frac{g(\lambda^k x)}{\lambda^{2k}} - \frac{g(\lambda^{k+1} x)}{\lambda^{2(k+1)}} \right\| \leqslant \frac{\varphi(\lambda^k x, \lambda^k x/n)}{(m+n)^2 \lambda^{2k}}. \tag{2.11}$$

Thus (2.8), (2.11) and the triangle inequality imply

$$\left\| g(x) - \frac{g(\lambda^{k+1} x)}{\lambda^{2(k+1)}} \right\| \leqslant \frac{1}{(m+n)^2} \sum_{i=0}^{k} \frac{\varphi(\lambda^i x, \lambda^i x/n)}{\lambda^{2i}}$$

for all $x \in X$, completing the proof of the required functional inequality (2.8) for $k+1$ by induction.

Step 2. We claim that the sequence $\{g_k(x)\}$ of mappings $g_k(x) := g(\lambda^k x)/\lambda^{2k}$ converges for all $x \in X$.

Note that from the completeness of Y one proves that the sequence is a Cauchy sequence in Y. In fact if $j > i > 0$, then

$$\|g_j(x) - g_i(x)\| = \left\| \frac{g(\lambda^j x)}{\lambda^{2j}} - \frac{g(\lambda^i x)}{\lambda^{2i}} \right\| = \frac{1}{\lambda^{2i}} \left\| \frac{g(\lambda^j x)}{\lambda^{2(j-i)}} - g(\lambda^i x) \right\|$$

holds for all $x \in X$. Applying the inequality (2.8) to the above with $x \to \lambda^i x$, we conclude that

$$\begin{aligned}
\|g_j(x) - g_i(x)\| &= \frac{1}{\lambda^{2i}} \left\| \frac{g(\lambda^{j-i} \lambda^i x)}{\lambda^{2(j-i)}} - g(\lambda^i x) \right\| \\
&\leqslant \frac{1}{\lambda^{2i}(m+n)^2} \sum_{l=0}^{j-i-1} \frac{\varphi(\lambda^{l+i} x, \lambda^{l+i} x/n)}{\lambda^{2l}} \\
&\leqslant \frac{1}{(m+n)^2} \sum_{l=i}^{j-1} \frac{\varphi(\lambda^l x, \lambda^l x/n)}{\lambda^{2l}} \\
&\to 0 \quad \text{as } i \to \infty,
\end{aligned}$$

which shows that $\{g_k(x)\}$ is a Cauchy sequence in Y.

Step 3. We conclude that the limit

$$Q(x) = \lim_{k\to\infty} \frac{g(\lambda^k x)}{\lambda^{2k}} = \lim_{k\to\infty} \frac{f(\lambda^k x)}{\lambda^{2k}}$$

exists for all $x \in X$ and $Q : X \to Y$ is a Pappus type quadratic mapping near the approximate Pappus type quadratic mapping $f : X \to Y$ with error (2.7).

For this purpose, one gets from Step 2 that Q is a well-defined mapping. In addition it is clear from (2.5) that the inequality

$$\frac{1}{\lambda^{2k}} \left\| D_{m,n} f(\lambda^k x, \lambda^k y) \right\| \leqslant \frac{1}{\lambda^{2k}} \varphi(\lambda^k x, \lambda^k y)$$

holds for all $x, y \in X$ and all $k \in \mathbb{N}$. Taking the limit $k \to \infty$, we see from the definition of Q that Q satisfies the equation

$$D_{m,n} Q(x, y) = 0,$$

that is, Q is a Pappus type quadratic mapping.

It follows easily from (2.8) and the definition of Q that the inequality (2.7) holds for all $x \in X$.

Step 4 (Proof of uniqueness). Let $\check{Q} : X \to Y$ be another Pappus type quadratic mapping satisfying the equation

$$D_{m,n} \check{Q}(x, y) = 0$$

and the approximate error bound

$$\left\| f(x) - \frac{nf(0)}{m + 2n} - \check{Q}(x) \right\| \leqslant \frac{1}{(m + n)^2} \Phi\left(x, \frac{x}{n}\right) \tag{2.12}$$

for all $x, y \in X$. To prove the aforementioned uniqueness we employ Eq. (2.1), so that

$$Q(x) = \lambda^{-2k} Q(\lambda^k x), \qquad \check{Q}(x) = \lambda^{-2k} \check{Q}(\lambda^k x)$$

hold for all $x \in X$ and all $k \in \mathbb{N}$. Thus the triangle inequality and inequalities (2.7), (2.12) yield the inequality

$$\begin{aligned} \left\| Q(x) - \check{Q}(x) \right\| &= \frac{1}{\lambda^{2k}} \left\| Q(\lambda^k x) - \check{Q}(\lambda^k x) \right\| \\ &\leqslant \frac{1}{\lambda^{2k}} \left(\left\| Q(\lambda^k x) + \frac{nf(0)}{m + 2n} - f(\lambda^k x) \right\| \right. \\ &\quad \left. + \left\| f(\lambda^k x) - \frac{nf(0)}{m + 2n} - \check{Q}(\lambda^k x) \right\| \right) \\ &\leqslant \frac{2}{(m + n)^2} \sum_{i=0}^{\infty} \frac{\varphi(\lambda^{i+k} x, \lambda^{i+k} x/n)}{\lambda^{2(i+k)}} \end{aligned}$$

for all $x \in X$ and all $k \in \mathbb{N}$. Therefore from $k \to \infty$, one establishes

$$Q(x) - \check{Q}(x) = 0$$

for all $x \in X$, completing the proof of uniqueness. The proof of Theorem 2.2 is complete. $\square$

Theorem 2.3. *Assume that a mapping $f : X \to Y$ satisfies*

$$\left\| D_{m,n} f(x, y) \right\| \leqslant \psi(x, y) \tag{2.13}$$

for all $x, y \in X$ with an approximate remainder $\psi : X^2 \to [0, \infty)$, and the series

$$\Psi(x, y) := \sum_{i=1}^{\infty} \lambda^{2i} \psi\left(\frac{x}{\lambda^i}, \frac{y}{\lambda^i}\right) \tag{2.14}$$

converges for all $x, y \in X$. Then there exists a unique Pappus type quadratic mapping $Q : X \to Y$ such that

$$D_{m,n} Q(x, y) = 0$$

for all $x, y \in X$ and

$$\left\| f(x) - Q(x) \right\| \leqslant \frac{1}{(m+n)^2} \Psi\left(x, \frac{x}{n}\right) \tag{2.15}$$

for all $x \in X$. The mapping Q is given by

$$Q(x) = \lim_{k \to \infty} \lambda^{2k} f\left(\frac{x}{\lambda^k}\right)$$

for all $x \in X$.

Proof. Observe that $\|m(m+n) f(0)\| \leqslant \psi(0, 0) = 0$ because of $\sum_{i=1}^{\infty} \lambda^{2i} \psi(0, 0) < \infty$ by hypothesis and hence $f(0) = 0$. From the same argument as that of (2.8)–(2.9), we obtain the crucial inequality

$$\left\| f(x) - \lambda^2 g\left(\frac{x}{\lambda}\right) \right\| \leqslant \frac{\lambda^2}{(m+n)^2} \psi\left(\frac{x}{\lambda}, \frac{x}{n\lambda}\right), \tag{2.16}$$

which induces similarly by induction

$$\left\| f(x) - \lambda^{2k} f\left(\frac{x}{\lambda^k}\right) \right\| \leqslant \frac{1}{(m+n)^2} \sum_{i=1}^{k} \lambda^{2i} \psi\left(\frac{x}{\lambda^i}, \frac{x}{n\lambda^i}\right) \tag{2.17}$$

for all $x \in X$ and all $k \in \mathbb{N}$.

Utilizing the last functional inequality (2.17) and the similar argument to Theorem 2.2, we can obtain the conclusion of this theorem. □

Corollary 2.4. *Let X be a normed space and Y a Banach space, and let θ, p and q be real numbers such that $0 \leqslant \theta$, either $p, q < 2$ or $p, q > 2$. Assume that a mapping $f : X \to Y$ satisfies*

$$\left\| D_{m,n} f(x, y) \right\| \leqslant \theta\left(\|x\|^p + \|y\|^q\right) \tag{2.18}$$

for all $x, y \in X$. Then there exists a unique Pappus type quadratic mapping $Q : X \to Y$ such that

$$D_{m,n} Q(x, y) = 0$$

for all $x, y \in X$ and

$$\left\| f(x) - \frac{nf(0)}{m+2n} - Q(x) \right\| \leqslant \frac{\theta}{n^2}\left(\frac{\|x\|^p}{|\lambda^p - \lambda^2|} + \frac{\|x\|^q}{n^q|\lambda^q - \lambda^2|} \right) \tag{2.19}$$

for all $x \in X$ ($x \in X \setminus \{0\}$ if $p, q \leqslant 0$), where $f(0) = 0$ if $p, q > 0$. The mapping Q is given by

$$\begin{cases} Q(x) = \lim_{n\to\infty} \frac{f(\lambda^k x)}{\lambda^{2k}} & \text{if } p, q < 2, \\ Q(x) = \lim_{k\to\infty} \lambda^{2k} f\left(\frac{x}{\lambda^k}\right) & \text{if } p, q > 2, \end{cases}$$

for all $x \in X$.

Proof. In case $p, q < 2$, we take account of $\varphi(x, y) := \theta(\|x\|^p + \|y\|^q)$ in Theorem 2.2. If $p, q > 2$, we consider $\psi(x, y) := \theta(\|x\|^p + \|y\|^q)$ in Theorem 2.3. If $p, q > 0$, by setting $x = y = 0$ in (2.18) one obtains $f(0) = 0$. □

In case $m = n = 1$, as a special case of Theorems 2.2 and 2.3 we have the Hyers–Ulam stability result for the quadratic functional equation (1.1) (see [3]).

Corollary 2.5. *Let X be a normed space and Y a Banach space, and let $0 \leqslant \theta$ be a real number. Assume that a mapping $f : X \to Y$ satisfies*

$$\left\| D_{m,n} f(x, y) \right\| \leqslant \theta$$

for all $x, y \in X$. there exists a unique Pappus type quadratic mapping $Q : X \to Y$ such that

$$D_{m,n} Q(x, y) = 0$$

for all $x, y \in X$ and

$$\left\| f(x) - \frac{nf(0)}{m+2n} - Q(x) \right\| \leqslant \frac{\theta}{m^2 + 2mn}$$

for all $x \in X$, where $\|f(0)\| \leqslant \theta/(m(m+n))$. The mapping Q is given by

$$Q(x) = \lim_{k\to\infty} \frac{f(\lambda^k x)}{\lambda^{2k}}$$

for all $x \in X$.

If $m = n = 1$ in the above corollary, one gets the result of Skof–Cholewa theorem [2].

3. Stability of (1.3) in Banach modules

In the last part of this paper, let B be a unital Banach algebra with norm $|\cdot|$, and let ${}_B\mathbb{M}_1$ and ${}_B\mathbb{M}_2$ be left Banach B-modules with norms $\|\cdot\|$ and $\|\cdot\|$, respectively.

As an application of the main Theorem 2.2 we are going to prove the generalized Hyers–Ulam stability problem of the functional equation (1.3) in Banach modules over a unital Banach algebra.

Theorem 3.1. *Assume that a mapping* $f : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ *satisfies*

$$\begin{aligned}\big\| D_{m,n,u} f(x,y) := {} & n^2 f(ux+muy) + mnf(ux-nuy) \\ & - (m+n)u^2\big(nf(x)+mf(ny)\big)\big\| \\ \leqslant {} & \varphi(ux,uy) \end{aligned} \tag{3.1}$$

for all $x, y \in {}_B\mathbb{M}_1$ *and all* $u \in B(|u| = 1)$ *with an approximate remainder* $\varphi : {}_B\mathbb{M}_1{}^2 \to [0,\infty)$, *and the series* (2.6) *converges for all* $x, y \in {}_B\mathbb{M}_1$. *If* f *is measurable or for each fixed* $x \in {}_B\mathbb{M}_1$, *the mapping* $t \to f(tx)$ *is continuous in* $t \in \mathbb{R}$, *then there exists a unique Pappus type quadratic mapping* $Q : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ *such that*

$$D_{m,n} Q(x,y) = 0 \quad \textit{and} \quad Q(bx) = b^2 Q(x)$$

for all $x, y \in {}_B\mathbb{M}_1$ *and all* $b \in B$, *and the inequality* (2.7) *for all* $x \in {}_B\mathbb{M}_1$.

Proof. By Theorem 2.2 it follows from (3.1) with $u = 1$ that there is a unique Pappus type quadratic mapping $Q : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ such that

$$D_{m,n} Q(x,y) = 0$$

for all $x, y \in {}_B\mathbb{M}_1$ and the inequality (2.7) for all $x \in {}_B\mathbb{M}_1$.

The mapping Q is given by

$$Q(x) = \lim_{k\to\infty} \frac{f(\lambda^k x)}{\lambda^{2k}}$$

for all $x \in {}_B\mathbb{M}_1$. Under the assumption that f is measurable or for each fixed $x \in {}_B\mathbb{M}_1$ $f(tx)$ is continuous in $t \in \mathbb{R}$, the mapping Q satisfies

$$Q(tx) = t^2 Q(x)$$

for all $x \in {}_B\mathbb{M}_1$ and for all $t \in \mathbb{R}$ [4]. Replacing x, y by $\lambda^k x, \lambda^k y$ in (3.1), respectively, and dividing it by λ^{2k}, we figure out

$$\frac{\|D_{m,n,u} f(\lambda^k x, \lambda^k y)\|}{\lambda^{2k}} \leqslant \frac{\varphi(\lambda^k ux, \lambda^k uy)}{\lambda^{2k}}$$

for all $u \in B$ $(|u| = 1)$ and for all $x, y \in {}_B\mathbb{M}_1$. Taking the limit $k \to \infty$, one obtains by condition (2.6) that

$$D_{m,n,u} Q(x,y) = 0 \tag{3.2}$$

for all $x, y \in {}_B\mathbb{M}_1$ and all $u \in B$ $(|u| = 1)$. Substituting x/n for y in (3.2) one gets that the equality

$$\begin{aligned} & n^2 Q(\lambda ux) - (m+n)^2 u^2 Q(x) = 0, \quad \text{or} \\ & Q(\lambda ux) - \lambda^2 u^2 Q(x) = 0, \quad \text{or} \\ & Q(ux) - u^2 Q(x) = 0 \end{aligned}$$

holds for all $x \in {}_B\mathbb{M}_1$ and all $u \in B$ $(|u| = 1)$. The last equality is also true for $u = 0$ vacuously. Now for each element $b \in B$ $(b \neq 0)$ we figure out

$$Q(bx) = Q\left(|b| \cdot \frac{b}{|b|}x\right) = |b|^2 \cdot Q\left(\frac{b}{|b|}x\right) = |b|^2 \cdot \frac{b^2}{|b|^2} \cdot Q(x) = b^2 Q(x)$$

for all $b \in B$ $(b \neq 0)$ and all $x \in {}_B\mathbb{M}_1$. Thus the mapping Q satisfies

$$Q(bx) = b^2 Q(x)$$

for all $b \in B$ and for all $x \in {}_B\mathbb{M}_1$, as desired. This completes the proof of the theorem. □

Since $\mathbb{C}$ is a Banach algebra, the Banach spaces M_1 and M_2 are considered as Banach modules over $\mathbb{C}$. Thus we have the following corollary.

Corollary 3.2. *Let φ be such mapping defined in Theorem* 3.1. *Let M_1 and M_2 be Banach spaces over the complex field $\mathbb{C}$. Suppose that a mapping $f : M_1 \to M_2$ satisfies*

$$\left\| D_{m,n,u} f(x, y) \right\| \leqslant \varphi(ux, uy)$$

for all $u \in \mathbb{C}$ $(|u| = 1)$ and for all $x, y \in M_1$. If f is measurable or for each fixed $x \in M_1$, the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique Pappus type quadratic mapping $Q : M_1 \to M_2$ such that

$$D_{m,n} Q(x, y) = 0 \quad \text{and} \quad Q(cx) = c^2 Q(x)$$

for all $x, y \in M_1$ and all $c \in \mathbb{C}$, and the inequality (2.7) *for all $x \in M_1$.*

Theorem 3.3. *Let φ be such mapping defined in Theorem* 3.1. *Assume that a mapping $f : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ satisfies*

$$\begin{aligned} &\left\| n^2u^2 f(x + my) + mnu^2 f(x - ny) - (m + n)\big(nf(ux) + mf(nuy)\big) \right\| \\ &\quad \leqslant \varphi(ux, uy) \end{aligned} \tag{3.3}$$

for all $u \in B$ $(|u| = 1)$ and for all $x, y \in {}_B\mathbb{M}_1$. If f is measurable or for each fixed $x \in {}_B\mathbb{M}_1$, the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique Pappus type quadratic mapping $Q : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ such that

$$D_{m,n} Q(x, y) = 0 \quad \text{and} \quad Q(bx) = b^2 Q(x)$$

for all $x, y \in {}_B\mathbb{M}_1$ and all $b \in B$, and the inequality (2.7) *for all $x \in {}_B\mathbb{M}_1$.*

Proof. The proof of this theorem is similar to that of Theorem 3.1. □

Remark 3.4. The following two analogous generalized Pappus' identities, (3.4) and (3.6) are established by the same manner as that of (1.2).

If $\triangle ABC$ is a triangle and E is the exterior point of the half-line $\overrightarrow{BC}$ with $n|\overline{CE}| = m|\overline{BE}|$, then $\triangle ABE$ is a triangle and C is the interior point of the side $\overline{BE}$ with $m|\overline{BC}| = (n - m)|\overline{EC}|$ and thus by the same argument as (1.2), the geometric identity

$$m^2|\overline{AB}|^2 + m(n - m)|\overline{AE}|^2 = mn|\overline{AC}|^2 + n(n - m)|\overline{CE}|^2 \tag{3.4}$$

holds for given positive integers $m, n \in \mathbb{N}$ $(n > m)$ (see Fig. 2).

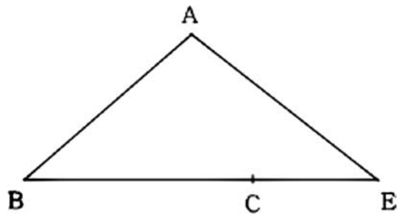

Fig. 2.

The above geometric identity yields the functional equation

$$m^2 Q\big(x+(n-m)y\big)+m(n-m)Q(x-my)=mnQ(x)+n(n-m)Q(my) \quad (3.5)$$

for a mapping $Q : X \to Y$ and for all $x, y \in X$ with X and Y linear spaces. Thus we obtain the generalized Hyers–Ulam stability problem for Eq. (3.5) by the same argument as that of Theorem 2.2 with $\lambda := n/m$.

On the other hand, if E is the exterior point of the half-line $\overrightarrow{CB}$ with $n|\overline{EB}| = m|\overline{EC}|$, then the geometric identity

$$(n-m)^2|\overline{AE}|^2 + m(n-m)|\overline{AC}|^2 = n(n-m)|\overline{AB}|^2 + mn|\overline{BC}|^2 \quad (3.6)$$

holds for given positive integers $m, n \in \mathbb{N}$ $(n > m)$. The above geometric identity yields similarly the functional equation

$$(n-m)^2 Q(x+my)+m(n-m)Q\big(x-(n-m)y\big)$$
$$= n(n-m)Q(x)+mnQ\big((n-m)y\big) \quad (3.7)$$

for a mapping $Q : X \to Y$ and for all $x, y \in X$ with X and Y linear spaces. Hence we get the generalized Hyers–Ulam stability problem for Eq. (3.7) by the same argument as that of Theorem 2.2 with $\lambda := n/(n-m)$.

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